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HYPERSURFACES WITH LOCALLY SYMMETRIC CONORMAL CONNECTION

Martin WIEHE

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Hypersurfaces with Locally Symmetric Conormal Connection

M. Wiehe*

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Abstract

This paper studies hypersurfaces admitting a locally symmetric connection which is induced by the Gauß conormal map in affine geometry. It is known that the rank of the shape operator is of importance to this topic. In dimension two we give new results for an arbitrary shape operator. In the case of a nondegenerate shape operator hypersurfaces with locally symmetric conormal connection can be treated as semi-Euclidean hypersurfaces. Moreover, we study whether and in which way a locally symmetric, projectively flat connection can be realized as the conormal connection of a hypersurface.

Keywords: Projectively flat connections, locally symmetric connections.
MOS-Classification: 53A15, 53B05, 53C35

0. Introduction

In Euclidean geometry the investigation of the Gauß map plays an important role for the classification of submanifolds. In affine hypersurface theory we have two Gauß maps, namely one induced by the transverse "normal" field, the second by the conormal bundle. It is well known that the Gauß conormal map induces a connection ∇^* which is projectively flat (see e.g. [11]). F. Dillen, K. Nomizu and L. Vrancken [1] proved that the projective flatness gives a geometric interpretation of the Gauß integrability conditions. This work stimulated further investigations of the properties of the Gauß conormal map (see e.g. [5]). In our paper we will investigate hypersurfaces for which the Gauß conormal map induces a locally symmetric connection.

In 1982, K. Nomizu [3] asked for more examples of affine hypersurfaces $f: M^n \to A^{n+1}$ with locally symmetric induced connection ∇, i.e. $\nabla R = 0$, where ∇ is the induced connection of a transversal field and R its curvature tensor. In 1985, P. Verheyen and L. Verstraelen [13] classified all regular hypersurfaces with Blaschke normal, $\nabla R = 0$ and $n \geq 3$. K. Nomizu and B. Opozda generalized this classification for arbitrary transversal fields in 1992, [4]. The classification of surfaces ($n = 2$) with $\nabla R = 0$ is much more complicate than in dimension $n > 2$. The case of these surfaces was investigated by E. Cartan, W. Slebodzinski and later by B. Opozda and W. Jelonek (see [7], [2]).

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The result of K. Nomizu and B. Opozda in [4] shows that for regular hypersurfaces with relative normal y and $n \geq 3$, the condition $\nabla R = 0$ implies $\nabla^* R^* = 0$. So the classification of hypersurfaces with locally symmetric conormal connection ∇^* generalizes the problem posed by K. Nomizu.

In section 2 we prove: In the case of a nondegenerate shape operator the condition $\nabla^* R^* = 0$ is equivalent to the fact that ∇ is a metric connection, i.e. there exists a symmetric, nondegenerate $(0,2)$-tensor g on M^n with $\nabla g = 0$. Regular hypersurfaces with metric connections were examined by K. Nomizu and B. Opozda in [5] and by B. Opozda in [8]. With their results we obtain for a regular hypersurface f with relative normal y and with $\nabla^* R^* = 0$ and nondegenerate shape operator:

(i) f is a semi-Euclidean hypersurface with semi-Euclidean normal y.

(ii) If the relative normal y is the centroaffine normal then f is an open part of a quadric with center.

(iii) If the relative normal y is the Blaschke normal then f has constant (semi-Euclidean) Gauß-curvature.

In section 2.2 the condition $\nabla^* R^* = 0$ is investigated for surfaces with degenerate shape operator. Proving a general result for conjugate connections, it turns out that the classification of equiaffine regular surfaces with $\nabla^* R^* = 0$ is equivalent to the known classification of surfaces with $\nabla R = 0$. If for a relative normal $\nabla R = 0$, then $(\nabla^* R^*)_p \neq 0$ in a point $p \in M^n$ if and only if y_p is not the equiaffine normal and the shape operator has rank one.

In section 3 we study the following realization problem: Given a manifold with a locally symmetric, projectively flat connection ∇^*. Can one immerse (M, ∇^*) as a regular hypersurface with relative normal such that the induced conormal connection is ∇^*. As a basic lemma for this we prove that a locally symmetric, projectively flat connection admits a symmetric Ricci-tensor.

1. Preliminaries

1.1. Affine hypersurfaces

For definitions and facts on affine differential geometry we refer the reader to [12]. Let M^n be a connected, orientable, n-dimensional C^∞-manifold with $n \geq 2$ and denote the $C^\infty(M^n)$-module of vector fields over M^n by $\mathfrak{X}(M^n)$. Let f be an immersion $f\colon M^n \to A^{n+1}$ into the $(n+1)$-dimensional real affine space A^{n+1} which is equipped with the canonical flat connection $\overline{\nabla}$.

The structure equations for an arbitrary transversal field y on $f(M^n)$ are:

$$\begin{array}{llcll} \text{Gauß-}f & \overline{\nabla}_u df(v) & = & df(\nabla_u v) + h(u,v)y\,, & \\ \text{Weingarten-}y & dy(v) & = & df(-Sv) + \theta(v)y & (u,v \in \mathfrak{X}(M^n)). \end{array}$$

Here, h is a symmetric $(0,2)$-tensor, ∇ a torsion-free connection, called the *induced connection*, S a $(1,1)$-tensor field, called the *shape operator*, and θ a one-form. Moreover, these coefficients are invariant under the change of $\{f, y\}$ to $\{f^\# = \alpha f, y^\# = L_\alpha y\}$, where α is an affine transformation of A^{n+1} and L_α the linear part of α.

In this paper only transversal fields with $\theta = 0$, called *relative normals*, are considered; we give examples in 1.3.

Since the curvature tensor $\overline{R}$ of $\overline{\nabla}$ is flat one derives the integrability conditions:

$$\begin{array}{llcl} \text{Gauß-}\nabla & R(u,v)w & = & h(v,w)Su - h(u,w)Sv, \\ \text{Codazzi-}h & (\nabla_u h)(v,w) & = & (\nabla_v h)(u,w), \\ \text{Codazzi-}S & (\nabla_u S)v & = & (\nabla_v S)u, \\ \text{Ricci-equation} & h(Sv,w) & = & h(v,Sw). \end{array}$$

For a relative normal y the essential part $y\colon M^n \to V^{n+1}$ of y is denoted by the same symbol. Here V^{n+1} is the vector space associated to A^{n+1}. If S is nondegenerate then one can prove that y is a hypersurface with (centroaffine) relative normal $(-y)$. In this case we get the structure equation:

$$\text{Gauß-}y \qquad \overline{\nabla}_u dy(v) = dy(\nabla'_u v) + h'(u,v)(-y)\,.$$

The relations of the coefficients with those of the structure equations of f are:

$$\begin{aligned} h'(u,v) &= h(Su,v)\,, && (1.1) \\ \nabla'_u v &= S^{-1}\nabla_u(Sv). && (1.2) \end{aligned}$$

For a hypersurface with relative normal y the associated *conormal field* $Y\colon M^n \to V_{n+1}$ on M^n is defined as the unique solution of:

$$Y(y) = 1 \qquad \text{and} \qquad Y(df(u)) = 0, \qquad (u \in \mathfrak{X}(M^n))\,, \qquad (1.3)$$

where V_{n+1} denotes the dual space of V^{n+1}. The pair $\{Y, y\}$ is called a *relative normalization.* If f is *regular* – i.e. h is nondegenerate – then $\operatorname{rank}(dY, Y) = n+1$ and we can consider Y as a hypersurface $Y\colon M \to V_{n+1}$ with transversal field $-Y$ and structure equation

$$\text{Gauß-}Y \qquad \overline{\nabla}_u dY(v) = dY(\nabla^*_u v) + \widehat{S}(u,v)(-Y).$$

∇^* is called the *conormal connection.* Furthermore we get the relations:

$$\begin{aligned} \widehat{S}(u,v) &= h(Su,v)\,, && (1.4) \\ xh(u,v) &= h(\nabla_x u, v) + h(u, \nabla^*_x v)\,. && (1.5) \end{aligned}$$

The integrability conditions for Y read:

$$\begin{array}{llcl} \text{Gauß-}\nabla^* & R^*(u,v)w & = & \widehat{S}(v,w)u - \widehat{S}(u,w)v, \\ \text{Codazzi-}\widehat{S} & (\nabla^*_v \widehat{S})(u,w) & = & (\nabla^*_u \widehat{S})(v,w). \end{array}$$

Hence, with the Gauß-∇^* equation, the Ricci-tensor Ric^* of ∇^* is given by:

$$\widehat{S} = \frac{1}{n-1}\mathrm{Ric}^*\,. \qquad (1.6)$$

1.2. Conjugate connections

If two torsion-free connections ∇ and $\widetilde{\nabla}$ on a semi-Riemannian manifold (M^n, h) satisfy

$$xh(u,v) = h(\nabla_x u, v) + h(u, \widetilde{\nabla}_x v)\,, \qquad (1.7)$$

they are called *conjugate* with respect to h and $\{\nabla, h, \widetilde{\nabla}\}$ is called a *conjugate triple.*

For a conjugate triple $\{\nabla, h, \widetilde{\nabla}\}$ define the tensors

$$C(u,v) := \frac{1}{2}(\nabla_u v - \widetilde{\nabla}_u v) \quad \text{and} \tag{1.8}$$

$$\widehat{T}(v) := \frac{1}{n}\,\mathrm{trace}\{u \mapsto C(u,v)\}\,. \tag{1.9}$$

If $\widehat{T} = 0$ then the triple (or the tensor C, resp.) is said to satisfy the *apolarity condition*. The curvature tensors R and $\widetilde{R}$ of ∇ and $\widetilde{\nabla}$ are related by (see [12], p. 56):

$$h(R(u,v)w,x) = -h(w,\widetilde{R}(u,v)x)\,. \tag{1.10}$$

1.3. Examples of relative normals

Let $f\colon M^n \to A^{n+1}$ be a regular hypersurface.
(i) The usual Euclidean normal is a relative normal. The invariance group of this relative normal is $O(n+1,\mathbb{R})$ plus translations.
(ii) Suppose that the position vector of f, denoted by the same symbol f, is never tangential. Then $y := -f$ is a relative normal, the so-called *centroaffine normal*. Here, the invariance group is $Gl(n+1,\mathbb{R})$.
(iii) There exists a relative normal – unique up to sign – such that the conjugate triple $\{\nabla, h, \nabla^*\}$ satisfies the apolarity condition. This relative normal has $Sl(n+1,\mathbb{R})$ plus translations as invariance group and is called the *equiaffine (or Blaschke) normal*.

1.4. Projectively flat connections

Two torsion-free connections ∇ and $\widetilde{\nabla}$ are called *projectively equivalent* if their pregeodesics coincide. A torsion-free connection ∇ is called *projectively flat* if it is locally projectively equivalent to a flat connection.
The *projective curvature tensor* W of Weyl is defined by:

$$W(u,v)w = R(u,v)w - [P(v,w)u - P(u,w)v] + [P(u,v) - P(v,u)]w\,, \tag{1.11}$$

$$P(u,v) = \frac{1}{n^2-1}(n\mathrm{Ric}(u,v) + \mathrm{Ric}(v,u))\,. \tag{1.12}$$

If the Ricci-tensor Ric of ∇ is symmetric then equation (1.11) reduces to:

$$W(u,v)w = R(u,v)w - \frac{1}{n-1}(\mathrm{Ric}(v,w)u - \mathrm{Ric}(u,w)v)\,. \tag{1.13}$$

If $\dim(M^n) > 2$ then ∇ is projectively flat if and only if $W = 0$. In dimension two, one has for arbitrary torsion-free connections

$$R(u,v)w = \mathrm{Ric}(v,w)u - \mathrm{Ric}(u,w)v \tag{1.14}$$

and therefore $W = 0$. In this case ∇ is projectively flat if and only if $(\nabla_u P)(v,w) = (\nabla_v P)(u,w)$.
Comparing these results with the integrability conditions Gauß-∇^*, Codazzi-$\widehat{S}$ and equation (1.6) we see that the conormal connection of a relative normalization is projectively flat. For details on projectively flat connections see e.g. the Appendix 1 of [6].

2. Locally symmetric conormal connection

The following theorem serves as a starting point of the investigation.

Theorem 2.1 [4]. *Let $n > 2$ and let $f\colon M^n \to A^{n+1}$ be a regular hypersurface with transversal field y. If the induced connection ∇ is locally symmetric, then either $S = 0$ on M^n or $f(M^n)$ is an open part of a quadric with center and ∇ is the equiaffine connection.*

An easy consequence (see e.g. the proof of Theorem 2.1 in [4]) is:

Corollary 2.2 *Let $n > 2$ and let $f\colon M^n \to A^{n+1}$ be a regular hypersurface with relative normalization $\{Y, y\}$. If $\nabla R = 0$ then $\nabla^* R^* = 0$.*

The proof of Theorem 2.1 in [4] is based on the evaluation of the equation $R(X,Y)R = 0$, where $R(X,Y)$ acts on R as a tensor derivation. This approach won't work for a locally symmetric conormal connection since one has in general:

Lemma 2.3 *If M^n is a manifold with a projectively flat, torsion-free connection ∇^* with symmetric Ricci-tensor, then $R^*(X,Y)R^* = 0$ for all $X, Y \in \mathfrak{X}(M^n)$.*

Proof. Just calculate $R^*(X,Y)R^*$ using (1.13), the assumption $W^* = 0$ and the symmetry of Ric^*. □

However, one gets the following description:

Proposition 2.4 *Let $f\colon M^n \to A^{n+1}$ be a regular hypersurface with relative normalization $\{Y, y\}$. The following conditions are equivalent:*

(i) $\nabla^* R^* = 0$,

(ii) $\nabla^* \widehat{S} = 0$,

(iii) $\nabla_u(Sv) = S\nabla^*_u v \qquad (u, v \in \mathfrak{X}(M^n))$.

Proof. The equivalence of (i) and (ii) follows from the Gauß-∇^* equation.
The equivalence of (ii) and (iii): With (1.1) and (1.2) we get:

$$(\nabla^*_u \widehat{S})(v, w) = h(\nabla_u(Sv) - S\nabla^*_u v, w)\ .$$

Since h is nondegenerate the proof is complete. □

Remark 2.5 For a regular hypersurface with $\nabla^* R^* = 0$ we get by Proposition 2.4 (iii) that $\ker(S)$ is ∇^*-parallel and $\mathrm{Im}(S)$ is ∇-parallel. Hence the rank of S is constant.

2.1. Nondegenerate shape operator

We obtain a more specific characterization of regular hypersurfaces with locally symmetric conormal connection if we assume that the shape operator is nondegenerate. The degenerate case is much more complicated and examined for surfaces in section 2.2. First we want to point out the following well known fact about locally symmetric connections which will be used in the proof of Proposition 2.7 below.

Lemma 2.6 *If a projectively flat connection ∇ on a semi-Riemannian manifold (M^n, g) is the metric connection of g, then ∇ is locally symmetric and (M^n, g) is of constant sectional curvature.*

Now we can prove the following characterization of regular hypersurfaces with locally symmetric conormal connection and nondegenerate shape operator. Note, that the equivalent statements (b.1), (b.2), (c.1), (c.3) and (c.4) were proved in [5], Proposition 3. We formulate the parts (b.1) and (b.2) weaker as in [5].

Proposition 2.7 *Let $f: M^n \to A^{n+1}$ be a regular hypersurface with a relative normalization $\{Y, y\}$. If S is nondegenerate, then the following properties are equivalent:*

(a.1) $\nabla^* R^* = 0$,

(b.1) ∇^* *is a metric connection,*

(b.2) ∇ *is a metric connection,*

(b.3) f *is a semi-Euclidean hypersurface with semi-Euclidean normal* y,

(c.1) $\nabla' = \nabla^*$,

(c.2) $\exists\, G \in Gl(V^{n+1}, V_{n+1})$ *with* $Y_p = G y_p \quad (p \in M^n)$,

(c.3) y *is an open part of a quadric with center at* $0 \in V^{n+1}$,

(c.4) Y *is an open part of a quadric with center at* $0 \in V_{n+1}$.

Proof. The implication (a.1) to (b.1) follows from Lemma 2.4 (ii) and (1.1) since $\widehat{S}$ is nondegenerate. The converse follows from Lemma 2.6. The equivalence of (b.1) and (b.2) is a well known fact for conjugate connections ([6], p. 159). Comparing Lemma 2.4 (iii) with (1.2) we get that (a.1) is equivalent to (c.1).
Suppose (c.1) is valid. Since the coefficients of the structure equations of y and Y coincide (see 1.4), Y and y differ by a centroaffine transformation (modulo duality). Hence (c.2) is satisfied. The converse is obvious.
The equivalence of (c.2) and (b.3): Suppose that (c.2) is valid. Define

$$B(X, Z) := (GX)(Z) \quad \text{for all} \quad X, Z \in V^{n+1}. \tag{2.1}$$

Since by (1.3)

$$B(y, y) = (Gy)(y) = Y(y) = 1 \tag{2.2}$$

we get with (1.3) and Weingarten–y:

$$0 = u(Gy)(y) = (Gdy(u))(y) + (Gy)(dy(u)) = -(Gdf(Su))(y) \qquad (u \in \mathfrak{X}(M^n)).$$

Since S is nondegenerate,

$$\begin{aligned} B(df(v), y) &= 0 \quad \text{and} & (2.3)\\ B(y, df(v)) &= Y(df(v)) = 0. & (2.4) \end{aligned}$$

Now we have with the Gauß-f equation

$$\begin{aligned} 0 &= vB(df(Su),y) = B(df(\nabla_v(Su)) + h(Su,v)y,y) - B(df(Su),df(Sv)) \\ &= \widehat{S}(u,v) - B(df(Su),df(Sv)) \,. \end{aligned}$$

Define now $g(u,v) := \widehat{S}(S^{-1}u, S^{-1}v) = B(df(u),df(v))$. With (2.2), (2.3), (2.4), the symmetry and non degeneracy of g we get the symmetry and non degeneracy of B. So B is a semi-Euclidean inner product on V^{n+1} and f is an isometric immersion with respect to g. With (2.2) and (2.3) we see that y is the semi-Euclidean normal.
The converse is obvious (see [12], p. 100).

The equivalence of (c.1) and (c.3) is an application of the generalisized theorem of Maschke in [12], p. 117 (the authors recently pointed out that the proof there is given for central quadrics only). Relation (1.3) shows that (c.3) is equivalent to (c.4). □

As a consequence of the proof we see that the index of the semi-Euclidean structure B (defined by (2.1)) of a regular hypersurface with locally symmetric conormal connection is uniquely determined by the connection. Therefore we get:

Corollary 2.8 *Let $f, f^{\#}: M^n \to A^{n+1}$ be regular hypersurfaces with relative normalizations. Suppose that $\nabla^* R^* = 0$, S is nondegenerate and $\nabla^{*\#} = \nabla^*$. For the semi-Euclidean inner products B and resp. $B^{\#}$ – defined by (2.1) – there exists an $L \in Gl(n+1,\mathbb{R})$ with $B^{\#}(LX, LZ) = B(X,Z)$ for all $X, Z \in V^{n+1}$.*

For a classification of hypersurfaces with $\nabla^* R^* = 0$ and nondegenerate shape operator the different normalizations must be investigated separately, since the property $\nabla^* R^* = 0$ can be realized on every regular hypersurface with the usual Euclidean normalization. Choosing $y = -f$ in Proposition 2.7 (c.3) we obtain for the centroaffine normalization:

Corollary 2.9 *Let $f: M^n \to A^{n+1}$ be a regular hypersurface with centroaffine normalization. Then, $\nabla^* R^* = 0$ if and only if $f(M^n)$ is an open part of a nondegenerate quadric with center at $0 \in V^{n+1}$.*

Thus from the three classical relative normalizations only the equiaffine normalization is left. For $n \geq 2$ we get (compare to [5], resp. [2] in dimension two):

Lemma 2.10 *Let $f: M^n \to A^{n+1}$ be a regular hypersurface with equiaffine normalization. Equivalent statements are:*

(i) $\nabla^* R^* = 0$ *and the affine* Gauß curvature $K_p := \det(S_p) \neq 0$ *for some point* $p \in M^n$,

(ii) *f is a semi-Euclidean hypersurface with constant Gauß curvature $\widetilde{K}$.*

Furthermore $K = \widetilde{K}$ in (i).

So globally one has (compare to [5]):

Corollary 2.11 *Let $f: M^n \to A^{n+1}$ be a hyperovaloid with equiaffine normalization $\{Y, y\}$, then*

$$\nabla^* R^* = 0 \iff (f, M^n) \quad \textit{is an hyperellipsoid} \,.$$

2.2. Locally symmetric conormal connections on surfaces

In this section we want to examine the special case of regular surfaces with locally symmetric conormal connection, including the case of a degenerate shape operator. (For the notation in the following result we refer to Section 1.2.)

Proposition 2.12 *Let (M^2, h) be a two-dimensional semi-Riemannian manifold and ∇, $\widetilde{\nabla}$ torsion-free connections that are conjugate with respect to h. Suppose ∇ is not flat. If $\nabla R = 0$ then $\widetilde{\nabla}\widetilde{R} = 0$ if and only if the conjugate triple $\{\nabla, h, \widetilde{\nabla}\}$ is apolar.*

Proof. Suppose that ∇ is not flat and $\nabla R = 0$. Using (1.10) and (1.8) we calculate:

$$\begin{aligned}
0 &= h((\nabla_z R)(u,v)w, x) \\
&= h(\nabla_z R(u,v)w, x) - h(R(\nabla_z u, v)w, x) - h(R(u, \nabla_z v)w, x) - h(R(u,v)\nabla_z w, x) \\
&= -z(w, \widetilde{R}(u,v)x) + h(w, \widetilde{R}(u,v)\widetilde{\nabla}_z x) \\
&\quad + h(w, \widetilde{R}(\nabla_z u, v)x) + h(w, \widetilde{R}(u, \nabla_z v)x) + h(\nabla_z w, \widetilde{R}(u,v)x) \\
&= -z(w, \widetilde{\nabla}_z \widetilde{R}(u,v)x) + h(w, \widetilde{R}(u,v)\widetilde{\nabla}_z x) \\
&\quad + h(w, \widetilde{R}(\nabla_z u, v)x) + h(w, \widetilde{R}(u, \nabla_z v)x) \\
&= -h(w, (\widetilde{\nabla}_z \widetilde{R})(u,v)x) + h(w, \widetilde{R}(2C(z,u), v)x) + h(w, \widetilde{R}(u, 2C(z,v))x) .
\end{aligned}$$

With $\dim(M^2) = 2$ and $\widehat{T}_r = \frac{1}{2}C^s_{rs}$ we obtain in local notation:

$$\begin{aligned}
(\widetilde{\nabla}_{\partial_r}\widetilde{R})(\partial_i, \partial_j)\partial_k &= \widetilde{R}(C(\partial_r, \partial_i), \partial_j)\partial_k + \widetilde{R}(\partial_i, C(\partial_r, \partial_j))\partial_k \\
&= C^s_{ri}\widetilde{R}(\partial_s, \partial_j)\partial_k + C^s_{rj}\widetilde{R}(\partial_i, \partial_s)\partial_k = 2\widehat{T}_r\widetilde{R}(\partial_i, \partial_j)\partial_k .
\end{aligned}$$

Since $\operatorname{Im}(R)$ is ∇-parallel and there exists a point $p \in M^2$ with $R_p \neq 0$ we obtain $R_p \neq 0$ for every point $p \in M^2$. From (1.10) we get $\widetilde{R}_p \neq 0$ for all $p \in M^2$. Hence $\widetilde{\nabla}\widetilde{R} = 0$ is equivalent to $\widehat{T} = 0$. □

Remark 2.13 For regular surfaces with equiaffine normalization, the condition $\nabla R = 0$ is equivalent to $\nabla^* R^* = 0$, since ∇ and ∇^* are conjugate with respect to h (see [2]).
So the classification of regular equiaffine normalized surfaces with $\nabla^* R^* = 0$ is equivalent to the classification of surfaces with $\nabla R = 0$, which is well known (see e.g. [7] and [2]).
Having in mind that, for $n > 2$, the condition $\nabla R = 0$ implies $\nabla^* R^* = 0$, one can state more precisely:

Proposition 2.14 *Let $f\colon M^2 \to A^3$ be a regular surface with a relative normalization $\{Y, y\}$ and locally symmetric induced connection ∇.*
There is a point $p \in M^2$ with $(\nabla^* R^*)_p \neq 0$ if and only if $\operatorname{rank}(S) = 1$ and y_p is not the equiaffine normal.

Proof. Suppose that $f\colon M^2 \to A^3$ is a regular surface and $\{Y, y\}$ a relative normalization with locally symmetric induced connection ∇. With the Gauß-f equation one can show (see [7]) that $\operatorname{Im}(R) = \operatorname{Im}(S)$, where $\operatorname{Im}(R_p) := \operatorname{span}\{R(u,v)w \mid u, v, w \in T_pM^2\}$ for $p \in M^2$. Since $\operatorname{Im}(R)$ is ∇-parallel for every manifold M^n with locally symmetric connection

∇ we get that $\mathrm{Im}(S)$ is ∇-parallel. So $\mathrm{rank}(S)$ is constant on M^2. Moreover we get with the conjugacy of ∇ and ∇^* – using (1.10) – that $\dim\mathrm{Im}(R_p^*) = \dim\mathrm{Im}(R_p) = \mathrm{rank}(S)$.
If $\mathrm{rank}(S) = 1$ then R is not flat. In this case Proposition 2.12 states that $T_p \neq 0$ implies $(\nabla^* R^*)_p \neq 0$ for $p \in M^2$. Hence we have done the "if"-part.
If $\mathrm{rank}(S) = 0$ we get with the Gauß-∇^* equation that $R^* = 0$, hence $\nabla^* R^* = 0$. Suppose $\mathrm{rank}(S) = 2$. To prove the "only if"-part we must show that $\nabla^* R^* = 0$. Ric is nondegenerate (on all of M^n) since one can prove with (1.14) that for arbitrary torsion-free connections on M^2 (see [7]): $\dim\mathrm{Im}(R_p) + \dim\ker(\mathrm{Ric}_p) = 2 \qquad (p \in M^2)$.
Since $\nabla R = 0$ we get with (1.14) that $\nabla\mathrm{Ric} = 0$. So ∇ is a metric connection with respect to Ric and Proposition 2.7 now states that $\nabla^* R^* = 0$ on M^2. □

Remark 2.15 There exist regular surfaces with the following properties (see [7]):
y is not the equiaffine normal on all of M^2, $\nabla R = 0$ and $\dim\mathrm{Im}(S) = 1$. Proposition 2.14 now states that for these surfaces: $\nabla^* R^* \neq 0$ on all of M^2.

3. An existence theorem

The fundamental theorem in affine hypersurface theory states:

Theorem 3.1 [1]. *Let (M^n, h) be a simply connected semi-Riemannian manifold with torsion-free connection ∇^*. Suppose the connection ∇^* admits a parallel volume form.*
Then there is a regular hypersurface $f\colon M^n \to A^{n+1}$ with relative normalization $\{Y, y\}$, such that ∇^* is the induced conormal connection and h the induced metric if and only if ∇^* is projectively flat and $\nabla^* h$ is totally symmetric.
Furthermore, the hypersurface f is unique up to an affine transformation of A^{n+1}.

The existence of a ∇^*-parallel volume form – required in Theorem 3.1 – is equivalent to the fact that the Ricci-tensor of ∇^* is symmetric (see [11], p. 99). First we want to examine if this condition is already satisfied for locally symmetric, projectively flat connections. In the two-dimensional case we have the following result which is proved in [9].

Lemma 3.2 *Let ∇ be a torsion-free connection on a two-dimensional manifold M. Equivalent statements are:*

(i) $\nabla\mathrm{Ric}$ *is totally symmetric,*

(ii) Ric *is symmetric and ∇ is projectively flat.*

Using equation (1.14) we have in particular that Ric is symmetric if $\nabla R = 0$. This result is valid for projectively flat connections in arbitrary dimension:

Lemma 3.3 *Let ∇ be a torsion-free and projectively flat connection on a manifold M. If ∇ is locally symmetric, then the Ricci-tensor of ∇ is symmetric.*

Proof. Let $\dim(M) \geq 3$ and consider the projective curvature tensor W of Weyl from (1.11) and (1.12). Since ∇ is projectively flat we have $W = 0$ and therefore

$$R(u,v)w = P(v,w)u - P(u,w)v - [P(u,v) - P(v,u)]w\ . \tag{3.1}$$

Since $\nabla R = 0$ we obtain for $u, v, w \in \mathfrak{X}(M)$

$$0 = (R(u,v)R)(u,v)w = [\cdots]u + [\cdots]v - 3(P(u,v) - P(v,u))^2 w$$

in a lengthy but straightforward calculation. Since we can choose $\{u, v, w\}$ linearly independent P is symmetric, which is equivalent to the symmetry of Ric:

$$0 = P(v,u) - P(u,v) = \frac{1}{n+1}[\mathrm{Ric}(u,v) - \mathrm{Ric}(v,u)] \,.$$

□

A first local consequence of Lemma 3.3 is

Theorem 3.4 *Let M^n be a manifold equipped with a projectively flat connection ∇^*. If ∇^* is locally symmetric then M^n can be immersed locally as a regular hypersurface such that ∇^* is the conormal connection of a relative normalization.*

Proof. By Lemma 3.3 the Ricci tensor of ∇^* is symmetric. Hence, to show the theorem by applying Theorem 3.1, we must show that there exists locally a metric h with totally symmetric covariant derivative.
It is known (see [10]) that for a projectively flat connection with symmetric Ricci tensor and for every function f on M^n the $(0,2)$-tensor field $H(f)(u,v) := \nabla^*_u \nabla^*_v f + \frac{1}{n-1} f \mathrm{Ric}^*(u,v)$ is symmetric and has totally symmetric covariant derivative. It is now easy to determine a local function f such that $H(f)$ is of maximal rank. Define now $h = H(f)$. □

The rank of a bilinear form B on M^n is defined for $p \in M^n$ by:

$$\begin{aligned} \mathrm{rank}(B_p) &:= n - \dim\ker(B_p)\,, \text{ with} \\ \ker(B_p) &:= \{v \in T_pM^n \mid B(v,u) = 0 \quad (u \in T_pM^n)\}. \end{aligned}$$

Theorem 3.5 *Let M^n be a simply connected manifold with a locally symmetric, projectively flat and torsion-free connection ∇^*. Suppose* $\mathrm{rank}_p(\mathrm{Ric}^*) = n$ *for some point $p \in M^n$.*
Then there exist a semi-Euclidean inner product B on A^{n+1}, such that, for every Ric^**-selfadjoint, nondegenerate $(1,1)$-field S with $\nabla^* S^{-1}$ totally symmetric, there is a regular, semi-Euclidean hypersurface $f\colon (M^n, g_B) \to (A^{n+1}, B)$ that induces S and ∇^*.*
The first fundamental form is given by $g_B(\cdot,\cdot) = \frac{1}{n-1}\mathrm{Ric}^*(S^{-1}(\cdot), S^{-1}(\cdot))$.

Proof. Let M^n be a simply connected manifold and ∇^* a locally symmetric, projectively flat connection on M^n. Assume there exist a point $p \in M^n$ with $\mathrm{rank}_p(\mathrm{Ric}^*) = n$.
Since ∇^* is projectively flat and locally symmetric, Ric^* is symmetric by Lemma 3.3. Hence R^* satisfies (see (1.13))

$$R^*(u,v)w = \frac{1}{n-1}[\mathrm{Ric}^*(v,w)u - \mathrm{Ric}^*(u,w)v] \qquad (u, v, w \in \mathfrak{X}(M^n))\,.$$

From $\nabla^* R^* = 0$ we get $\nabla^* \mathrm{Ric}^* = 0$. Therefore $\mathrm{rank}(\mathrm{Ric}^*) = n$ on all of M^n, since $\ker(\mathrm{Ric}^*)$ is ∇^*-parallel. So $\widehat{S}(u,v) := \frac{1}{n-1}\mathrm{Ric}^*(u,v)$ is a metric on M^n. Let S be a

$(1, 1)$-field satisfying the conditions posed in the theorem. Define now $h(u, v) := \widehat{S}(u, S^{-1}v)$. h is a semi-Riemannian metric. We get with $\nabla^*\widehat{S} = 0$ and the total symmetry of ∇^*S^{-1} that

$$\begin{aligned}(\nabla^*_u h)(v, w) &= u\widehat{S}(v, S^{-1}w) - \widehat{S}(\nabla^*_u v, S^{-1}w) - \widehat{S}(v, S^{-1}\nabla^*_u w) \\ &= \widehat{S}(v, (\nabla^*_u S^{-1})w) = \widehat{S}(v, (\nabla^*_w S^{-1})u) \\ &= (\nabla^*_w h)(v, u) \ .\end{aligned}$$

So $\nabla^* h$ is totally symmetric and there exists a regular hypersurface $\widetilde{f}: M^n \to A^{n+1}$ and a relative normalization $\{\widetilde{Y}, \widetilde{y}\}$ that induces h as induced metric and ∇^* as conormal connection. Since ∇^* is metric, there exist by Proposition 2.7 a semi-Euclidean inner product B_S on A^{n+1}, such that $\widetilde{f}$ is a semi-Euclidean immersion with semi-Euclidean normal $\widetilde{y}$.

Now we must show that there exist one semi-Euclidean inner product B for every possible choice of S. Since id meets the conditions on S, construct $B := B_{id}$ by the foregoing scheme. With Corollary 2.8 there exists $L \in Gl(n+1, \mathbb{R})$ with $B(u, v) = B_S(Lu, Lv)$. Define $f := L^{-1}\widetilde{f}$ and $y := L^{-1}\widetilde{y}$. Because of the affine invariance of the coefficients of the structure equations we obtain $h = h(y)$ and $\nabla^* = \nabla(Y)^*$ for the normalization $\{Y, y\}$. Since obviously y is orthogonal to f with respect to B and $B(y, y) = 1$, f is a semi-Euclidean hypersurface with respect to B. □

Remark 3.6 The Codazzi condition between ∇^* and S^{-1}, i.e. ∇^*S^{-1} is totally symmetric, can be replaced by the assumption that the connection $\nabla := (S\nabla^*)S^{-1}$ is torsion-free.

If $\text{rank}(\text{Ric}^*_p) = n$ for some point $p \in M^n$, then (M^n, Ric^*) is, under the assumptions of Lemma 3.3, a semi-Riemannian manifold with metric connection ∇^* and constant sectional curvature. As expected one gets:

Corollary 3.7 *Let M^n be a simply connected manifold with a locally symmetric, projectively flat and torsion-free connection ∇^* with $\text{rank}_p(\text{Ric}^*) = n$ for some point $p \in M^n$. Then, M^n can be immersed into a semi-Euclidean space (V^{n+1}, B) as an open part of a quadric centered at $0 \in V^{n+1}$, such that $\frac{1}{n-1}\text{Ric}^*$ is the first fundamental form and ∇^* its metric connection.*

Proof. Choosing $S = id$ in Theorem 3.5, the normal y is the centroaffine normal (see the structure equation Weingarten-y). Use now Corollary 2.9. □

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